ON THE ATTENUATION OF SHOCK-WAVES IN A MOVING MEDIUM WITH VARYING DENSITY AND TEMPERATURE

(O ZATUKHANII UDARNYKH VOLN V DVIZHUSHCHEISIA SREDE S PEREMENNYMI PLOTNOST'IU I TEMPERATUROI)

PMM Vol. 24, No. 5, 1960, pp. 912-915

O. Iu. POLIANSKII
(Moscow)

(Received 10 May 1960)

Many papers are devoted to the problem of the nonlinear attenuation of the shock-waves in a homogeneous medium, for example [1-4].

The propagation of shock-waves of small amplitude in an inhomogeneous medium was considered by Gubkin [5], who applied for this purpose the method of integration of the equations along the characteristics.

Below, the problem of the propagation of shock-waves in a moving medium with varying density and temperature is considered under the assumption that the shock-wave is weak and the wavelength is much smaller than the characteristic dimension of the problem. It seems that nonlinear effects in this case may be determined using an approach analogous to the one applied by Landau in [1]. Asymptotic formulas for excess pressure at the wave-front are given for linear pressure profiles and certain laws of variation of the density of a medium.

In the case of a "weakly varying medium", simple approximate formulas are obtained which allow the excess pressure at the wave-front with linear pressure profile to be calculated.

1. We shall consider the propagation of weak shock-waves in an inhomogeneous medium. We assume that the undisturbed pressure p, density ρ, the velocity of sound a and the velocity of gas motion u (wind velocity) are functions of the coordinates. We shall also assume that in the whole region under consideration

$$\frac{\Delta p}{p} \ll 1, \qquad \frac{l}{H} \ll 1, \qquad \frac{l}{R} \ll 1 \tag{1.1}$$

where Δp is the excess pressure in the wave, l is the wavelength (in what follows we shall consider l to be the length of the compression zone of the wave), H is the characteristic dimension of the problem, i.e. the distance in which the parameters of the medium substantially vary, R is the radius of curvature of the wave-front.

From the last inequality (1.1) it follows that cylindrical and spherical waves in any small region may be considered to be locally plane. We shall introduce the Cartesian coordinate system x, y, z.

We shall assume that the solutions of the equations of motion in the approximation of geometric acoustics are known, i.e. the configuration of the trajectories and the variation of the excess pressure $\Delta p'$ and the wavelength l' along the ray (primed values mean that the values of the corresponding parameters are taken in the approximation of geometrical acoustics).

It is proposed to establish the additional attenuation not accounted for by acoustics of a shock wave with a linear pressure profile along an arbitrary chosen trajectory $\xi(x, y, z)$.

As the basis of the considerations below will be taken the properties of one-dimensional travelling waves, which are determined by Riemann's solutions of the equations of motion (so-called simple waves; see, for example, [6], Section 94), according to which the translational velocity of a point of the wave-profile U is equal to the sum of the local sound velocity and the gas velocity in the wave v or, in a slightly different form,

$$U = a + \alpha v \qquad \left(\alpha = \frac{\varkappa + 1}{2}, \ \varkappa = \frac{c_p}{c_v}\right) \tag{1.2}$$

where the quantity α is given in brackets for an ideal gas; κ is the ratio of specific heats for constant pressure and constant volume (for air α = 1.2).

For a shock Riemann's solution loses its validity; however, for waves of small amplitude, up to the terms of the second order with respect to $\Delta p/p$, the wave remains simple, and the location of the discontinuity is found by the use of a simple rule, described in [6], Section 95.

According to (1.2) an additional translation of the profile element of the wave which occurs along the trajectory in the direction of the normal to the wave-front in the time from t_0 to t is determined by the expression

$$\delta\xi = \alpha \int_{t_0}^{t} v dt$$

where, in first approximation

$$v = \Delta p' / \rho a, \quad dt = U_*^{-1} d\xi;$$

U_* is the so-called ray velocity, equal to $|a\mathbf{n} + \mathbf{u}|$, where $\mathbf{n}$ is a unit vector, directed along the normal to the wave front and $d\xi$ is the differential of an arc along the trajectory. We have

$$\delta\xi = \alpha \int_{\xi_0}^{\xi} \frac{\Delta p'}{\rho a} \frac{d\xi}{U_*}, \qquad U_* = a \sqrt{1 + \frac{2u_n}{a} + \left(\frac{u}{a}\right)^2} \qquad (u_n = \mathbf{u} \cdot \mathbf{n}) \tag{1.3}$$

where ξ_0 is the location of an element of the wave-profile at the time t_0 on the trajectory under study; hereafter we shall denote by ξ_θ the location of the shock-wave for $t = t_0$).

Note that the wavelength in the approximation of geometric acoustics is determined by the expression

$$l'(t) = l_0 \frac{N(t)}{N(t_0)} \qquad \left(l_0 = l(t_0), \ \frac{N(t)}{N(t_0)} = \frac{N(\xi)}{N(\xi_0)} = \frac{a(\xi) + u_n(\xi)}{a(\xi) + u_n(\xi_0)} \right) \tag{1.4}$$

Taking into consideration (1.3) and (1.4) and repeating, essentially, the arguments of Landau [6], Section 95, we obtain the following expressions for the wavelength l and the excess pressure at the wave-front Δp_Φ (for the wave with linear pressure profile):

$$l(\xi) = l_0 \frac{N(\xi)}{N(\xi_0)} \sqrt{1 + \Phi(\xi, \xi_0)}, \quad \Phi(\xi, \xi_0) = \frac{\alpha}{l_0} \frac{N(\xi_0)}{N(\xi)} \int_{\xi_0}^{\xi} \frac{\Delta p_\Phi'}{\rho a} \frac{d\xi}{U_*} \tag{1.5}$$

$$\Delta p_\Phi(\xi) = \frac{\Delta p_\Phi'}{\sqrt{1 + \Phi(\xi, \xi_0)}} \tag{1.6}$$

2. Let us investigate the asymptotic behavior of shock-waves in an isothermal medium at rest with varying density. In this case the trajectories will be straight lines, and

$$\Delta p_\Phi'(\xi) = \frac{\Delta p_0}{(\xi/\xi_0)^\nu} \sqrt{\frac{\rho}{\rho_0}} \qquad (\Delta p_0 = \Delta p_\Phi'(\xi_0), \ \rho_0 = \rho(\xi_0))$$

where $\nu = 0$, $1/2$, 1, corresponding to the case of plane cylindrical and spherical waves. Then for (1.6) we obtain

$$\Delta p_\Phi = \frac{\Delta p_0}{(\xi/\xi_0)^\nu} \sqrt{\frac{\rho}{\rho_0}} \frac{1}{\sqrt{1 + m_\nu \Psi(\xi, \xi_0)}} \tag{2.1}$$

where

$$m_\nu = \frac{\alpha \Delta p_0 \xi_0}{l_0 a_0^2 \rho_0}, \qquad \Psi(\xi, \xi_0) = \int_{\xi_0}^{\xi} \frac{d\xi}{(\xi/\xi_0)^\nu \xi_0 \sqrt{\rho/\rho_0}}$$

If $\sqrt{\rho}$ increases along the trajectory faster than $\xi^{1-\nu}$, then the particular law of attenuation of the intensity of a shock-wave will be

$$\Delta p_{\Phi} \sim \frac{\sqrt{\rho}}{\xi^{\nu}} \tag{2.2}$$

i.e. the same as in acoustics.

It is to be noted that to fulfil the condition (1.1) the density must not increase along the trajectory faster than exponentially.

If ρ = const, then from (2.1) we obtain the asymptotic formulas of Landau [1]:

$$\begin{aligned} \Delta p &\sim \frac{1}{\sqrt{\xi}} && \text{for } \nu = 0 \\ \Delta p &\sim \frac{1}{\xi^{3/4}} && \text{for } \nu = 1/2 \\ \Delta p &\sim \frac{1}{\xi\sqrt{\ln(\xi/\xi^*)}} && \text{for } \nu = 1 \end{aligned} \tag{2.3}$$

If $\sqrt{\rho}$ along a trajectory decreases (or increases, but not faster than $\xi^{1-\nu}$), then formally it is not difficult to obtain the asymptotic laws of the attenuation of a shock-wave in this case also. However, the requirement that the inequalities (1.1) be fulfilled imposes a substantial limitation upon the laws of density variation, for which an asymptotic representation for Δp_{Φ} is possible within the limits of the approximation considered.

In the case of $\sqrt{\rho} \sim \xi^{c-\nu}$, where $-1 < c < 1$, i.e. if ρ decreases not faster than ξ^{-4}, then in the limit

$$l \sim \xi^{\frac{1-c}{2}}, \quad \Delta p_{\Phi} \sim \rho^{3/4} \xi^{-\frac{1+\nu}{2}} \tag{2.4}$$

In this case, if Δp_{Φ} is represented in the form

$$\Delta p_{\Phi} \sim \Delta p_{+} \sqrt{\frac{\rho}{\rho_0}}\, \mu(\rho)$$

where Δp_{+} is the excess pressure at the wave-front propagating in a homogeneous medium (see Formulas (2.3)), then for $\xi \to \infty$ the following equations are valid:

$$\begin{aligned} \mu(\rho) &= (\rho/\rho_0)^{1/4} && \text{for } \nu = 0 \text{ and } \nu = 1/2 \\ \mu(\rho) &= (\rho/\rho_0)^{1/4} \sqrt{\ln(\xi/\xi^*)} && \text{for } \nu = 1 \end{aligned}$$

The function $\mu(\rho)$ characterizes the nonlinear attenuation of the shock-wave intensity, determined by the variation of the density of the medium.

3. We shall investigate a medium in which all the hydrodynamic parameters in the unperturbed condition depend only on one coordinate, for example z. Assume that in the entire region under consideration

$$\frac{|\operatorname{grad} a|}{a_0}(\xi - \xi_0) < \varepsilon \ll 1, \quad \frac{|\operatorname{grad} u|}{a_0}(\xi - \xi_0) < \varepsilon \ll 1$$

In that case, up to terms of order ϵ, the quantity $\Delta p'$ along a trajectory may be represented in the form

$$\Delta p' = \Delta p_0 \sqrt{\frac{\rho}{\rho_0}} \sqrt{\frac{a}{a_0}} \sqrt{\frac{s_0}{s}} \left(1 + \frac{u_n - u_{n0}}{a_0}\right)^{-1} \tag{3.1}$$

where the index 0 denotes the values of the flow parameters in some chosen cross-section 0 of an infinitely narrow stream tube constructed around the given trajectory and s is the area of cross-section of such a stream tube. Formula (3.1) may be proved in the following manner. From [5] we have

$$\Delta p_\Phi' = \frac{\Delta p_0}{L} \sqrt{\frac{\rho}{\rho_0}} \sqrt{\frac{a}{a_0}}$$

where in the case under consideration

$$L = \exp\left[\frac{1}{2}\int_0^t \left(a \operatorname{div} \mathbf{n} + \frac{du_x}{dz} n_x n_z + \frac{du_y}{dz} n_y n_z\right) dt\right]$$

(integration is along the trajectory). Since div $\mathbf{u}/a = 0$ and $dt = d\xi/U_*$, then up to terms of order ϵ we have

$$L = \exp\left\{\frac{1}{2}\int_{\xi_0}^{\xi} \frac{1}{a + u_n}\left[(a + u_n)\operatorname{div}\left(\mathbf{n} + \frac{\mathbf{u}}{a}\right) - u_n \operatorname{div} \mathbf{n} + \frac{du_n}{d\xi}\right] d\xi\right\}$$

$$= \exp\left\{\frac{1}{2}\int_{\xi_0}^{\xi} \frac{ds}{s} + 2\frac{du_n}{a_0}\right\} = \sqrt{\frac{s}{s_0}}\left[1 + \frac{u_n - u_{n0}}{a_0} + O(\varepsilon^2)\right]$$

Note that Formula (3.1) up to terms of order ϵ coincides with the formula for Δp_Φ given in [7] and obtained from the condition of conservation of average energy in geometric acoustics for stationary processes. The divergence of the stream tube s/s_0 is not difficult to obtain using the law of deflection of a ray in an inhomogeneous medium, as developed by Chibisov [8].

For linear variation of sound velocity in the absence of wind, we have* for a wave propagating from a spatial point source a first approximation $\sqrt{s} \sim \xi\sqrt{a}$, so that

$$\Delta p_\Phi' = \frac{\Delta p_0}{\xi/\xi_0}\sqrt{\frac{\rho}{\rho_0}}, \quad \Delta p_\Phi = \frac{\Delta p_\Phi'}{\sqrt{1+m\chi(\xi,\xi_0)}} \qquad \chi(\xi,\xi_0) = \frac{a_0}{a}\int_{\xi_0}^{\xi}\frac{d\xi}{\xi\sqrt{\rho/\rho_0}\,(a/a_0)^2} \tag{3.2}$$

As is known [4], for $\xi/\xi > 1.06$ for

$$\Delta p_0 = 0.57\,p, \quad m = 2.75, \qquad \xi_0 = \sqrt[3]{\frac{E_0}{p}} \quad (E_0 \text{ is the energy of explosion})$$

the results, obtained from the formula of Landau [1], which is Formula (3.2) when ρ and a are constants, coincide with the results of numerical calculation for a point explosion when the back pressure is taken into account [10-12].

Therefore, for an approximate calculation of Δp_Φ in a weakly varying medium with a constant gradient of sound velocity for $\xi \gg \xi_0$, the following formula may be suggested:

$$\Delta p_\Phi = \frac{0.57 p_0}{\xi/\xi_0}\sqrt{\frac{\rho}{\rho_0}}\frac{1}{\sqrt{1+2.75\chi(\xi\xi_0)}}$$

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Translated by J.R.W.